GROUP THEORY (BMath-II, Final Exam)

Answer as many questions as you like, for a maximum score of 50. Total time 3 *hours.*

You may use any result covered in the class without proving it. Use only results and notions covered in the course.

---

1. Let G be a finite group and N a normal subgroup of G. Prove that there is a composition series of G which has N as one of its terms. (8)

2. Compute the center Z of U(2) and prove that $\mathrm{U}(2) = \mathrm{SU}(2).\mathrm{Z}$. (8+8)

3. Let p and q be distinct primes and G be a non-abelian group of order pq. Prove that G is solvable but not nilpotent. (8)

4. Show that the additive group $(\mathbb{Q}, +)$ of rationals cannot be written as a semi-direct product of two proper subgroups. (8)

5. Investigate if the dihedral group D_{110} of order 220 is nilpotent. (10)

6. Is it true that two groups having composition series with isomorphic terms are isomorphic? Justify your answer. (8)

7. Prove that a solvable group G has a composition series if and only if it is finite. (10)

8. Determine all group homomorphisms $\mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^{\times}$, the multiplicative group of $\mathbb{C}$. (10)

9. Let $\mathrm{su}(2) = \{\mathrm{A} \in \mathrm{M}_2(\mathbb{C}),\ \mathrm{A}^* = -\mathrm{A}\}$, here $A^* = \overline{A}^t$. **(i)** Show that su(2) is a vector space over $\mathbb{R}$ with matrix addition and usual scalar multiplication from $\mathbb{R}$. **(ii)** Compute the dimension of su(2) as an $\mathbb{R}$-vector space. **(iii)** Show that, for $A, B \in \mathrm{su}(2)$, $[\mathrm{A}, \mathrm{B}] \in \mathrm{su}(2)$, where $[A, B] := AB - BA$. (5+8+5)

10. Show that the exponential map $\exp : \mathrm{M}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ maps su(2) onto SU(2). (10)